# Basis-Sensitive Quantum Typing via Realizability

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FunLeP

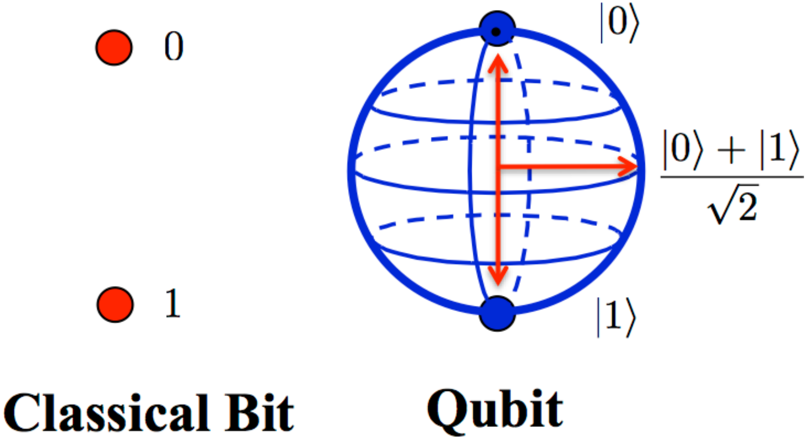
# Demistifying Quantum Computation

*A classical computer, makes use of physical phenomena (for example voltage on a wire) to represent information (bits) and perform computations.*

# Demistifying Quantum Computation

*A quantum computer, makes use of quantum physical phenomena (for example polarization on a photon) to represent information (quantum bits or qubits for short) and perform computations.*

# How do we represent qubits?



# Remistifying Quantum Computation

- No-cloning theorem

- No-deletion theorem

- Unitary operations

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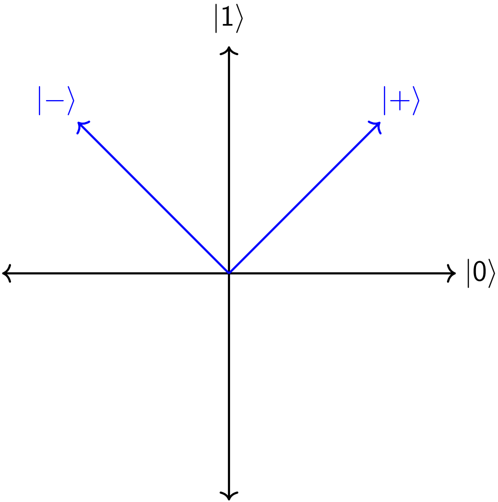
- Unitary operations

$$\lambda x.\lambda y.x \text{ XOR } y$$

# Shift the frame of reference

No cloning theorem states that we cannot copy *arbitrary* qubits.



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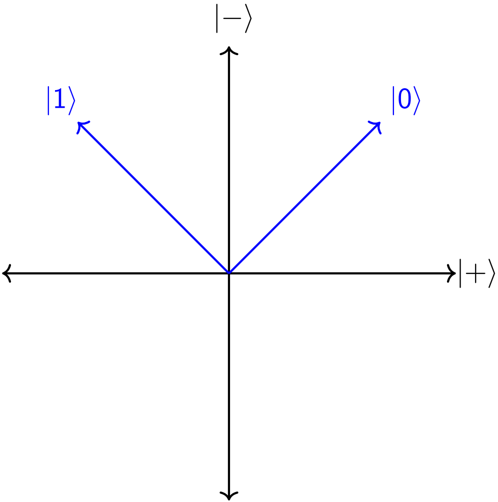

# Realizability technique

1. Define a machine (calculus w/reduction strategy)
2. Types are sets of values in the language
3. $\Gamma \vdash t : A$ is valid when $t\langle\sigma\rangle \to^* v \in A$.
   Typing rules become theorems we must prove

$$\frac{\Delta \vdash r : B}{\Gamma \vdash t : A}$$

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Safety properties now become trivial. The work is in proving the validity of the rules

# The $\lambda_B$ machine[1]

| | | | | |
|---|---|---|---|---|
| **Pure values** | $v$ | $::=$ | $x \mid \lambda x^B . \vec{t} \mid (v, v) \mid \lvert 0\rangle \mid \lvert 1\rangle$ | |
| **Pure terms** | $t$ | $::=$ | $w \mid t\,t \mid \mathtt{let}\ (x^B, y^B) = \vec{t}\ \mathtt{in}\ \vec{t} \mid (t, t)$ | |
| | | | $\mid \mathsf{case}\ \vec{t}\ \mathsf{of}\{\vec{v} \mapsto \vec{t} \mid \cdots \mid \vec{v} \mapsto \vec{t}\}$ | |
| **Value distributions** | $\vec{v}$ | $::=$ | $v \mid \vec{v} + \vec{v} \mid \alpha \cdot \vec{v}$ | $(\alpha \in \mathbb{C})$ |
| **Term distributions** | $\vec{t}$ | $::=$ | $t \mid \vec{t} + \vec{t} \mid \alpha \cdot \vec{t}$ | $(\alpha \in \mathbb{C})$ |

[1] *Based on "Realizability in the unitary sphere"* A. Díaz-Caro, M. Guillermo, A. Miquel, B. Valiron. LICS 2019

# Reduction strategy: call-by-basis

Let $B = \{\vec{b_1}, \cdots, \vec{b_n}\}$ and $\vec{w} \equiv \sum_{i=1}^{n} \alpha_i \vec{b_i}$. Then:

$$(\lambda x^B . \vec{t_i})\, \vec{w} \to \sum_{i=1}^{n} \alpha_i \vec{t_i}\ [\vec{b_i}/x]$$

# Examples

$$(\lambda x^B \,.\, \vec{t_i})\vec{w} \to \sum_{i=1}^{n} \alpha_i \vec{t_i}\ [\vec{b_i}/x]$$

Let $\mathbb{Z} = \{|0\rangle, |1\rangle\}$ and $\mathbb{X} = \{\overbrace{\frac{|0\rangle + |1\rangle}{\sqrt{2}}}^{|+\rangle}, \overbrace{\frac{|0\rangle - |1\rangle}{\sqrt{2}}}^{|-\rangle}\}$:

$$(\lambda x^{\mathbb{Z}} \,.\, (x, x))\ |0\rangle \to$$

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$$(\lambda x^{\mathbb{Z}} \,.\, (x, x))\ |+\rangle \to \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad (\lambda x^{\mathbb{X}} \,.\, (x, x))\ |0\rangle \to \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

# Types defined as sets

$$[\![\flat_X]\!] := X \qquad \text{(Where: } X \text{ is an orthonormal basis)}$$

$$[\![A \times B]\!] := \big\{(\vec{v}, \vec{w}) : \vec{v} \in [\![A]\!],\ \vec{w} \in [\![B]\!]\big\}$$

$$[\![A \Rightarrow B]\!] := \{\sum_{i=1}^{n} \alpha_i(\lambda x^B\,.\,\vec{t_i}) \in \mathcal{S}_1 \ :\ \forall \vec{w} \in [\![A]\!], \\ (\sum_{i=1}^{n} \alpha_i(\lambda x^B\,.\,\vec{t_i}))\vec{w} \Vdash B\}$$

$$[\![\sharp A]\!] := ([\![A]\!]^{\perp})^{\perp}$$

$$\text{Where: } A^{\perp} = \{\vec{v} \in \mathcal{S}_1 \mid \langle \vec{v} \mid a\rangle = 0, \forall a \in A\}$$

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# Subtyping

$$A \leq B \iff [\![A]\!] \subseteq [\![B]\!]$$

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$$\frac{\mathrm{size}(X) = \mathrm{size}(Y)}{\sharp(\flat_X) \cong \sharp(\flat_Y)} \qquad \frac{}{\flat_{X \times Y} \cong \flat_X \times \flat_Y}$$

$$\frac{A \leq B}{\sharp A \leq \sharp B} \qquad \frac{}{(\sharp A) \times (\sharp B) \leq \sharp(A \times B)}$$

# Type System

$$\frac{\Gamma \vdash \vec{s} : A \Rightarrow B \quad \Delta \vdash \vec{t} : A}{\Gamma, \Delta \vdash \vec{s}\,\vec{t} : B} \text{ (App)}$$

$$\frac{\Gamma, x : A \vdash \sum_{i=1}^{n} \alpha_i \vec{t_i} : B \quad A \leq \sharp\flat_X}{\Gamma \vdash \sum_{i=1}^{n} \alpha_i \lambda x^{\flat_X} . \vec{t_i} : A \Rightarrow B} \text{ (UnitLam)}$$

# Characterization of unitary operators

**Lemma**

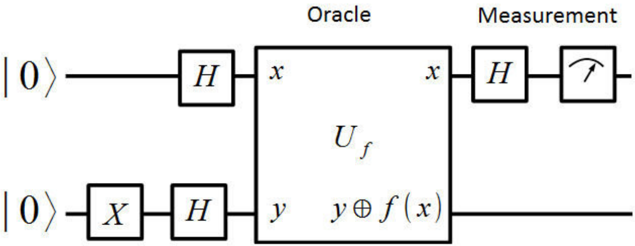
*$\lambda x^X . \vec{t} \in [\![\sharp\flat_X \Rightarrow \sharp\flat_Y]\!]$ iff exists $\vec{w}_i \in [\![\sharp\flat_Y]\!]$ such that $\forall \vec{v}_i \in [\![\flat_X]\!]$:*

$$\vec{t}\,[\vec{v}_i/x] \twoheadrightarrow \vec{w}_i \perp \vec{w}_j \twoheadleftarrow \vec{t}\,[\vec{v}_j/x] \qquad \text{if } i \neq j$$

**Theorem**

*$\vdash (\lambda x^X . \vec{t}) : \sharp\flat_X \Rightarrow \sharp\flat_Y$ iff it represents a unitary operator $F : \mathbb{C}^n \to \mathbb{C}^n$.*

## Use case: Deutsch's algorithm



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$$\text{Deutsch}_{\mathbb{Z}} :: (\sharp\flat_{\mathbb{Z}} \Rightarrow \sharp\flat_{\mathbb{Z}} \Rightarrow \sharp\flat_{\mathbb{Z}} \times \sharp\flat_{\mathbb{Z}}) \Rightarrow \sharp\flat_{\mathbb{Z}} \times \sharp\flat_{\mathbb{Z}}$$

$$\text{Deutsch}_{\mathbb{Z}} := \lambda f \,.\, \text{let } (x, y) = (f\,(\mathbb{H}\,|0\rangle)\,(\mathbb{H}\,|1\rangle)) \text{ in } (\mathbb{H}\,x, y).$$

# Use case: Deutsch's algorithm

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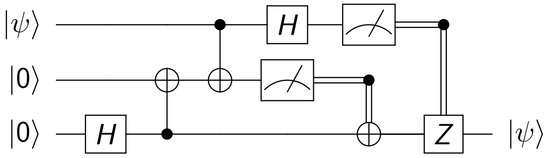
# Use case: Teleportation protocol




$$\text{Bell} = \left\{\frac{|00\rangle + |11\rangle}{\sqrt{2}}, \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \frac{|01\rangle - |10\rangle}{\sqrt{2}}\right\}$$

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$$\begin{aligned}\text{Teleport}_{\Phi^+} := \lambda x^{\mathbb{Z}}.\,\texttt{let}\ (y_1^{\mathbb{Z}}, y_2^{\mathbb{Z}}) = \Phi^+\ \texttt{in}\ \text{case}\ (x, y_1)\ \text{of}\ \{ & \\ & {\color{red}\Phi^+} \mapsto (\Phi^+, y_2), \\ & {\color{red}\Phi^-} \mapsto (\Phi^-, Zy_2), \\ & {\color{red}\Psi^+} \mapsto (\Psi^+, Xy_2), \\ & {\color{red}\Psi^-} \mapsto (\Psi^-, ZXy_2)\}.\end{aligned}$$

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$$
\begin{aligned}
\text{Teleport} := \lambda b^{\text{Bell}}\,.\,\lambda x^{\mathbb{Z}}\,.\,&\texttt{let}\ (y_1^{\mathbb{Z}}, y_2^{\mathbb{Z}}) = b\ \texttt{in}\\
&\text{let}(c_1, c_2, c_3, c_4) = (\text{corrections}\ \ b)\ \text{in}\\
&\text{case}\ (x, y_1)\ \text{of}\ \{\\
&{\color{red}\Phi^+} \mapsto (\Phi^+, c_1\ \ y_2),\\
&{\color{red}\Phi^-} \mapsto (\Phi^-, c_2\ \ y_2),\\
&{\color{red}\Psi^+} \mapsto (\Psi^+, c_3\ \ y_2),\\
&{\color{red}\Psi^-} \mapsto (\Psi^-, c_4\ \ y_2)\}.
\end{aligned}
$$

## Closing remarks

- Extend an existing calculus to work with multiple bases
- Realizability gives a flexible framework to define typing rules with different bases
- These new typing rules assist in the characterization of unitary operators
- The extended syntax allow for more versatile representations of algorithms.

*Bonus Track*

## Limits of this system: Entanglement

**Definition**

An entangled quantum system is a system which cannot be written as the tensor product of two sub-systems. In short, the behaviour of a particle cannot be independently described from the others.

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# Specification system

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{x : |+\rangle \vdash x : |+\rangle}\ (\mathsf{Ax})
        \quad
        \dfrac{}{y : |+\rangle \vdash y : |+\rangle}\ (\mathsf{Ax})
      }{x : |+\rangle, y : |+\rangle \vdash (x, y) : |+\rangle \times |+\rangle}\ (\mathsf{Pair})
    }{x : |+\rangle \vdash (x, x) : |+\rangle \times |+\rangle}\ (\mathsf{Contr})
  }{\vdash \lambda x^{|+\rangle}.(x, x) : |+\rangle \to (|+\rangle \times |+\rangle)}\ (\mathsf{Lam})
  \quad
  \dfrac{}{\vdash |+\rangle : |+\rangle}\ (\mathsf{Ax})
}{\vdash (\lambda x^{|+\rangle}.(x, x))|+\rangle : |+\rangle \times |+\rangle}\ (\mathsf{App})
$$